PROGRESS IN FREE ASSOCIATIVE ALGEBRAS[†]

BY

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CONTENTS

1. Introduction

This survey consists essentially of two parts. In the first place it reports on progress in the study of free algebras since the appearance of the author's book *Free Rings and their Relations* (Cohn [71^b]); this occupies Sections 2–7 of this survey and it also includes some topics (such as language theory) not touched on in that book. The second part, Sections 8–10, consists of notes on what might be called "non-commutative algebraic geometry". The remarkable development of algebraic geometry in recent years, which in principle makes the subject co-extensive with the theory of commutative rings, leads one to believe that there must be such a thing as "non-commutative algebraic geometry", a view clearly shared by others (cf. Amitsur [66], Graev-Kirillov [68], p. 376, Bergman [70], Shatz [72], p. 86, Taylor [73]). Since the publication of Cohn [71^b], it has become a little clearer to the author how one might set about developing such a project, and in particular, what the problems are that must be solved at the

[†] To the memory of J. Levitzki

outset. All that is intended here is to describe some of these problems, their connexion with free algebras, and the consequences which their solution would entail. It is already clear that even if all the problems mentioned here were solved, we would be faced with new questions. But this is as it should be, and although one cannot expect non-commutative algebraic geometry to be in any way a close counterpart of the commutative branch, the latter is of some help in suggesting lines of enquiry, and the author is hopeful that it will eventually develop into an independent field of study with its own flavour.

Since knowledge of free associative algebras — beyond the definition — is not very widespread, it seemed advisable to include enough information to allow this survey to be read independently of other sources, such as Cohn [71^b]. In fact, the early parts can to some extent be used as an introduction to that work. Proofs have normally been omitted, except for occasional outlines and brief deductions. In addition a number of open problems are included.

The second part, as indicated, is less a report than a programme. In Section 8 the analysis of homomorphisms into skew fields given in Cohn [71^b] Chapt. 7 is briefly described, and some of its consequences are listed; some of the latter appear here for the first time (for example, the simple construction of field extensions of different left and right dimensions). Section 9 discusses the problem of solving equations over skew fields; it may be regarded as an analysis of the problem: when does an equation have a solution? The fact that this basic question is still unanswered shows the primitive state of the subject; all we are able to offer here is a reduction to other, possibly more tractable problems. The final Section 10 describes some definitions of algebraically closed skew fields, and the formulation of such results as the Hilbert Nullstellensatz.

Except where otherwise stated, all rings have a unit-element which is inherited by subrings, preserved by homomorphisms, and acts unitally on modules. By a *field* we understand a not necessarily commutative division ring; the prefix "skew" is sometimes used for emphasis. The set of non-zero elements in an integral domain R is written $R^{\times}$.

2. The basic properties recalled

i) Let k be a commutative field and X any set; we shall denote the free k-algebra on X by $k\langle X\rangle$. Its elements may be uniquely expressed in the form

$$\Sigma a_w w, \tag{1}$$

where w runs over X^*, the free monoid on X, and $a_w \in k$ (but $a_w = 0$ for almost all w). Thus $k\langle X\rangle$ may also be regarded as the monoid algebra $k[X^*]$ on X^* over k. When X consists of a single element x, the free algebra is just the commutative polynomial ring $k[x]$, but as soon as X has more than one element, $k\langle X\rangle$ is non-commutative. The number of elements of X is determined by the isomorphism type of $k\langle X\rangle$; it is called the *rank* of $k\langle X\rangle$ (Cohn [69^a]).

An important property of $k\langle X\rangle$ is the fact that the generating set X is right linearly independent over the algebra itself. Thus, taking X finite for simplicity, say $X = \{x_1, \cdots, x_d\}$, we can write every element of $k\langle X\rangle$ uniquely in the form

$$a = \alpha + \Sigma x_i a_i, \text{ where } \alpha \in k,\ a_i \in k\langle X\rangle.$$

Suitably formulated, this property characterizes free algebras. In fact a stronger independence property holds in the graded ring associated with $k\langle X\rangle$. It is known as the "weak" algorithm since it is a generalization of the familiar Euclidean algorithm, to which it reduces in the commutative case.

ii) By a *filtration* on a ring R, one understands a function $v(x)$ from $R^\times$ $(= R - \{0\})$ to the integers, while $v(0) = -\infty$ by convention, with the properties

V.1. $v(x) \geqq 0$ for $x \neq 0$,

V.2. $v(x - y) \leqq \max\{v(x), v(y)\}$,

V.3. $v(xy) \leqq v(x) + v(y)$,

V.4. $v(1) = 0$.

For example, the usual degree on $k\langle X\rangle$, defined for the element a given by (1) as $\max\{l(w) \mid a_w \neq 0\}$ (where $l(w)$ is the length of the word w), is a filtration. We note that for this function equality holds in V.3. Generally we shall refer to the values of a filtration as *degrees*. Observe that every ring has the *trivial* filtration

$$v(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ -\infty & \text{if } x = 0. \end{cases}$$

Let R be a ring with a filtration v; then a family (a_i) of elements of R is said to be *right v-dependent* if some $a_i = 0$, or if there exist elements $b_i \in R$, almost all 0, such that

$$v(\Sigma a_i b_i) < \max_i\{v(a_i) + v(b_i)\}.$$

Otherwise (a_i) is *right v-independent*. An element $a \in R$ is said to be *right v-dependent* on a family (a_i) if $a = 0$ or if there exist elements $b_i \in R$, almost all 0, such that

$$v(a_i) + v(b_i) \leqq v(a) \text{ for all } i, \text{ and } v(a - \Sigma a_i b_i) < v(a).$$

The notion of *left* v-dependence is defined correspondingly.

It is clear that if a member of a family is right v-dependent on the rest, then that family is right v-dependent. The converse need not hold; for example, for the trivial filtration on R it holds if and only if R is a skew field, and even for more general filtrations the converse imposes strong restrictions on R:

DEFINITION. A ring R with filtration v is said to satisfy *n-term weak algorithm* (WA_n for short) relative to v if, given any right v-dependent family $a_1, \cdots, a_m$ $(m \leqq n)$, such that $v(a_1) \leqq \cdots \leqq v(a_m)$, some a_i is right v-dependent on $a_1, \cdots, a_{i-1}$. If R satisfies WA_n for all n, we say that R satisfies the *weak algorithm* (WA) relative to v.

The condition WA_n on a filtered ring is left-right symmetric, and the free algebra $k\langle X\rangle$ satisfies WA relative to the usual degree-function; in fact one has the following characterization of free algebras (Bergman [67], Cohn [61^a, 69^a 71^b]):

THEOREM 2.1. *Let R be an algebra over a commutative field k, with a positive filtration v such that $v(x) = 0$ if and only if $x \in k^\times$. Then R is the free associative k-algebra on a right v-independent free generating set if and only if R satisfies the weak algorithm relative to v. The filtration is determined by its values on a free generating set, where it may assume arbitrary positive values.*

The weak algorithm can also be defined relative to partially ordered monoids (cf. Bergman [67]); for a development, including a study of filtered modules, see Dicks [74].

iii) The universal mapping property of $k\langle X\rangle$ may be described as a substitution property: any mapping of X into a k-algebra A extends to a unique homomorphism $k\langle X\rangle \to A$, that is, we can substitute any elements of A for the elements of X. But one may want this property for K-rings, where K is a (possibly skew) field. We recall that for any ring Λ, a Λ-*ring* R is a ring which is also a Λ-bimodule such that $(xy)z = x(yz)$ for any x, y, z from Λ or R. Since R has a 1, we can also describe it by saying that the mapping $\alpha \mapsto \alpha.1$ is a ring-homomorphism $\Lambda \to R$, and conversely, any such homomorphism provides a Λ-ring structure on R.

Let K be a skew field; to form a free K-ring over a set X, where the elements of X need not commute with the elements of K, we shall have to specify the elements that commute with the elements of K. They will form a subfield k of K, and moreover, k must lie in the centre of K if we are to be able to substitute arbitrary elements of K for X. Thus we are given a skew field K which is a k-algebra; then the free K-ring on X over k is the free product

$$F = K_k^* k\langle X\rangle. \tag{2}$$

This ring has an evident universal mapping property for mappings from X to K-rings which are k-algebras; moreover, relative to the usual X-degree it again satisfies the weak algorithm. This follows easily by regarding F as the tensor K-ring on the K-bimodule ${}^N(K^{op} \otimes_k K)$, where $N = |X|$ and ${}^N V$ denotes the direct sum of N copies of V. This time, however, there are many rings other than the free rings satisfying the weak algorithm (that is, the straightforward analogue of Theorem 2.1 does not hold), essentially because the ring $K^{op} \otimes_k K$ is generally far from semisimple (cf. Bergman [67], Cohn [71^b]). For a study of the ring (2 when K/k is a finite-dimensional central simple algebra, see Procesi [68].

iv) It is well known, and easily proved, that a commutative domain with Euclidean algorithm is a principal ideal domain. A corresponding statement holds for rings with weak algorithm, with free ideal rings in place of principal ideal domains. By a *free right ideal ring*, or *right fir* for short, one understands a ring R in which each right ideal is free, as R-module, of uniquely determined rank. *Left* firs are defined similarly and a left and right fir is called simply a *fir*.

Any right fir is an integral domain; if it is also right Noetherian, it is necessarily a principal right ideal domain. Thus apart from a rather special case, firs are not Noetherian. Nevertheless they always satisfy a certain chain condition: for any $n \geqq 1$, every ascending chain of right ideals on at most n generators in a right fir becomes stationary (Cohn [71^b]). More briefly, we say that such rings satisfy right ACC_n. Left ACC_n is defined similarly.

THEOREM 2.2. (Cohn [63^b, 71^b], Bergman [67]). *Any filtered ring with weak algorithm is a fir; hence it satisfies left and right* ACC_n *for all* n.

In the study of WA_n a wider class of rings is useful. Let $n \geqq 1$; by an *n-fir* we understand a ring in which every right ideal on at most n generators is free, of unique rank. This condition is always left-right symmetric (unlike the notion of right fir, cf. Skornyakov [65], Bergman [67] and Cohn [71^b]. A 1-fir is just an integral domain, a 2-fir is an integral domain R with the property that $aR + bR$ is principal whenever $aR \cap bR \neq \emptyset$, and as n increases, the class of n-firs becomes smaller until we come to the class of rings that are n-firs for all n;

such a ring is called a *semifir*. Clearly every left or right fir is a semifir. A simple example of a semifir that is not a fir is a non-Noetherian Bezout ring (even commutative, for example $k[\mathbf{Q}^+]$, the monoid algebra of the positive rational numbers).

THEOREM 2.3 (Cohn [69[b], 71[b]]). *Any filtered ring with* WA_n *is an n-fir with left and right* ACC_n.

v) Unique factorization can be defined for non-commutative rings, an early example being the ring of linear differential operators. We shall not repeat the definition here (cf. Cohn [63[a], 71[b], 73[a]]), but merely recall that any atomic 2-fir (that is, where every element not zero or a unit can be written as a product of unfactorable elements) is a unique factorization domain. This includes all rings with WA_2 and in particular, all free algebras.

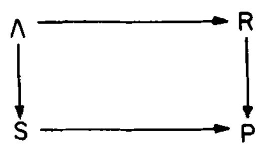

Fig. 1

vi) An important principle of construction in free ring theory is the free product. If R, S are any Λ-rings, their coproduct P in the category of Λ-rings always exists, and may be described by the push-out diagram shown in Fig 1. Now assume that R, S are *strict* Λ-rings, that is, the mappings from Λ to R and S are injective. The coproduct P is called *faithful* in case R, S are embedded in P and *separating* when the diagram is also a pull-back. A faithful separating coproduct of strict Λ-rings R, S is called the *free product* of R, S over Λ. Whereas the coproduct always exists, it need not be a free product. A sufficient condition is that R, S be faithfully flat as right Λ-modules (Cohn [59]). Thus the free product always exists over a skew field, and more generally, over any semisimple ring (cf. also Bergman [74[b]]).

Any free product of skew fields (over a common subfield) is a fir (Cohn [60, 63[b]], Bergman [74[a]]. In fact this result accounts for the origin of the notions of fir and weak algorithm. The result has been considerably generalized by Bergman:

THEOREM 2.4 (Bergman [74[b]]). *Let R be the free product of a family* (R_λ) *of strict K-rings, where K is semisimple. Then*

(i) *the right global dimension of R is given by*

$$\text{r. gl. dim } R = \begin{cases} \sup (\text{r. gl. dim } R_\lambda) & \textit{if this is positive,} \\ 0 \textit{ or } 1 & \textit{if each } R_\lambda \textit{ is semisimple;} \end{cases}$$

(ii) *every projective R-module M has the form* $M = \oplus M_\lambda \otimes R$, *where* M_λ *is a projective* R_λ*-module and the tensor product is taken over* R_λ, *while the direct sum is over the different* λ.

Part (i) has also been obtained by S. M. Gersten [74]. Theorem 2.4 provides very precise information on the free product. Thus it shows that R is a (semi) fir whenever all the factors R_λ are, and it leads to a proof of the following theorem.

THEOREM 2.5 (Cohn [64^b], Bergman [67]). *The group ring of a free group over a commutative field is a fir.*

For another method of obtaining a part of this result (by localization) see Section 8.

3. Lie and Jordan structures

i) From any k-algebra A two other k-algebra structures A^-, A^+ may be derived, not necessarily associative (or with unit-element), even if A is. Both have the same underlying k-module as A, but with multiplication

(1) $$[x, y] = xy - yx \text{ in } A^-,$$

(2) $$\langle x, y \rangle = xy + yx \text{ in } A^+.$$

Both these notions are of importance in their own right, but with close connexions to associative algebras, and we shall henceforth assume A to be associative. As is well known, the subalgebras of algebras A^-, where A is associative, are *Lie algebras*, that is, they satisfy the laws

$$[x, x] = 0, \ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \text{ (Jacobi-identity).}$$

With each abstract Lie algebra L one associates a universal associative enveloping algebra U_L. This is an associative algebra with a representation of L in U_L, that is, a Lie algebra homomorphism $\lambda: L \to U_L^-$, which is universal for such representations of L in associative algebras. Moreover, for Lie algebras over a field, λ is injective. In particular, if we take the free Lie algebra on a set X and form its universal associative envelope, we obtain $k\langle X \rangle$, the free associative algebra on X. It follows that the Lie subalgebra L_X of $k\langle X \rangle^-$ generated by X is free on X cf., e.g., Cohn [65].

The elements of L_X are frequently called the *Lie elements* in $k\langle X\rangle$; to characterize them we introduce a coalgebra structure on $F = k\langle X\rangle$. The mapping

$$\Delta: x \mapsto x \otimes 1 + 1 \otimes x \qquad (x \in X)$$

extends to a unique homomorphism of F into $F \otimes F$, the *comultiplication*, and with this definition, and the usual augmentation, F is easily seen to be a Hopf algebra (Sweedler [69]). An element c of F is said to be *primitive* if

$$\Delta(c) = c \otimes 1 + 1 \otimes c.$$

The set P of all primitive elements of F is always a Lie algebra, and in case k has finite characteristic p, P is a restricted Lie algebra (cf. Jacobson [62]). Now the Lie elements in F may be characterized by the fact that P is the restricted Lie algebra generated by X. For characteristic 0 this means that an element of F is a Lie element if and only if it is primitive (this is Friedrichs' criterion, cf. Cohn [54^b], Jacobson [62]).

ii) Similarly the subalgebras of A^+, where A is associative, are *Jordan algebras*, that is, they satisfy the laws

$$xy = yx, \qquad (xy)y^2 = (xy^2)y. \tag{3}$$

Moreover, even over a field, not every Jordan algebra can be embedded in an algebra of the form A^+, where A is associative (it would be unreasonable to expect this, because the second equation in (3) has degree four, whereas the associative law, like the Jacobi-identity, has degree three).

As before, we can now define the *Jordan elements* of F as the elements of the subalgebra of F^+ generated by X, but they are harder to characterize. No general test analogous to Friedrichs' criterion is known, and we confine ourselves to giving some special results.

In any free algebra $k\langle X\rangle$ there is an involution (that is, an anti-automorphism of period two) which is linear and leaves each $x \in X$ fixed. It is called *reversal*, and the elements fixed by it are said to be *reversible*, for example, *abba*.

THEOREM 3.1 (Cohn [54^a, 65],7). *Let k be a field of characteristic not two. Then an element of $k\langle x_1, x_2, x_3\rangle$ is a Jordan element if and only if it is reversible.*

For more than three generators Theorem 3.1 is false: the element $x_1x_2x_3x_4 + x_4x_3x_2x_1$ is reversible, but is mapped to an element of degree 4 in the exterior algebra on $x_1, \cdots, x_4$, whereas every Jordan element is mapped to a linear element.

This example easily leads to another necessary condition for Jordan elements. Of course it is enough to consider homogeneous elements.

Let f be a homogeneous element of degree n in $k\langle X\rangle$. If f is a Jordan element, then it is annihilated by the operator $\Sigma(\operatorname{sgn}\sigma)\,\sigma$ *where σ runs over the symmetric group on n letters, and* sgn σ *is the alternating character.*

By combining this result with Theorem 3.1 it is possible to obtain other necessary conditions, but no necessary and sufficient conditions for Jordan elements are known. A result on the structure of left-normed elements has recently been obtained by Robbins. A product is *left normed* if all brackets begin on the left of the first argument, for example, $(xy)\,(zt)$ is left normed in x, y, zt, but not in x, y, z, t. In particular, we may now speak of left-normed Lie products (or Jordan products).

THEOREM 3.2 (Robbins [71]). *The left-normed Jordan products in X and the left-normed Jordan products in the Lie elements of odd degree in X span the same space.*

iii) The characterization of Lie elements given earlier has a well-known analogue for group elements. Let H be the group algebra on the free group X, and define a comultiplication by

$$\Delta(x) = x \otimes x \qquad (x \in X). \tag{3}$$

This defines a Hopf algebra structure on H, and the free group can be recovered as the set of elements satisfying condition (3) (the "group-like" elements).

There is another coalgebra structure, involving coaddition as well as comultiplication. This has been used to give a categorical characterization of the polynomial ring $k[x]$, by Clark-Bergman [73].

4. Automorphisms and subalgebras

i) Let $F = k\langle X\rangle$ be a free k-algebra. Any automorphism of F is completely determined by its effect on the free generating set X. Moreover, any mapping $\alpha\colon X \to F$ can be extended to a unique endomorphism of F, which is an automorphism if and only if the family $X^\alpha = \{x^\alpha \mid x \in X\}$ generates F freely.

We recall that F has an *augmentation*, that is, a k-algebra homomorphism $\varepsilon\colon F \to k$, obtained by mapping each polynomial f in the x's to its constant term $\varepsilon(f)$. The kernel $\mathfrak{a}$ of ε is called the *augmentation ideal*; of course it depends on the choice of free generating set X. An automorphism which maps $\mathfrak{a}$ into itself is said to be *augmentation-preserving*. Further, a mapping of the form

$$x \mapsto x + a_x \qquad (x \in X,\ a_x \in k)$$

gives rise to an automorphism called a *translation* (relative to X). Now any automorphism α can be factored thus:

$$x \mapsto x + \varepsilon(x^\alpha) \mapsto x^\alpha,$$

which shows that any automorphism can be written as a translation followed by an augmentation-preserving automorphism.

Any free algebra $k\langle X\rangle$ has certain elementary automorphisms:

(a) *replace an element x of X by λx, where $\lambda \in k^\times$;*

(b) *permute the elements of X;*

(c) *replace a single $x \in X$ by $x + f(x_1, \cdots, x_r)$, where f is an expression in the elements $x_1, \cdots, x_r$ of X distinct from x, and leave all elements other than x in X unchanged.*

Here X should be thought of as ordered, or at least indexed. Thus $x_1 \mapsto x_2$, $x_2 \mapsto x_1$ is an instance of (b), and it defines a non-identity automorphism of $k\langle x_1, x_2\rangle$.

An automorphism of $F = k\langle X\rangle$ which can be expressed as a product of elementary automorphisms (that is, of the form (a)-(c)), is called *tame* (relative to X), all others are *wild*. Corresponding definitions may be given for polynomial rings, Lie algebras, groups, etc. For a free group of finite rank, Nielsen proved in 1924 that all automorphisms are tame (cf. Magnus-Karrass-Solitar [66]), and for polynomial rings in two variables Jung [42] proved the same. The corresponding result for free algebras of rank two is due to Makar-Limanov [70] and Czerniakiewicz [71, 72]:

THEOREM 4.1. *Every automorphism of the free algebra $k\langle x, y\rangle$ is tame.*

The two proofs are quite different, although both rely on Jung's theorem. There is a natural mapping $\mathrm{Ab}\colon k\langle X\rangle \to k[X]$ from the free algebra to the polynomial ring, which consists in making the algebra commutative, and which may be called *abelianization*. It induces a homomorphism of automorphism groups

$$\mathrm{Aut}(k\langle X\rangle) \to \mathrm{Aut}(k[X]), \tag{1}$$

and Czerniakiewicz proves Theorem 4.1 by showing that when X has two elements, (1) is injective; by Jung's theorem it is actually an isomorphism.

Whether all automorphisms of a free algebra of rank greater than two are tame is still open. An example by Bergman (cf. Czerniakiewicz [72]) shows that the free algebra on x, y over an integral domain Λ not a field has wild automorphisms (which he constructs as a tame automorphism over the field of fractions of Λ).

Theorems 3.1 and 4.1 combine to show that in $k\langle x, y\rangle$ the Jordan algebra generated by x and y is independent of the choice of x and y. For it consists of the set of reversible elements and in the 2-variable case this is mapped into itself by every tame automorphism; by Theorem 4.1 there are no others.

D. R. Lane [74] has proved that for any algebraically closed field k, any automorphism of $k\langle x, y\rangle$ is conjugate either to an augmentation-preserving automorphism or to an elementary automorphism and a scalar multiplication. A corresponding result follows for the polynomial ring $k[x, y]$.

It is an easy consequence of Theorem 4.1 that any automorphism of $k\langle x, y\rangle$ leaves the commutator $[x, y] = xy - yx$ unchanged except for a constant (non-zero) factor. The converse is open, thus the question is as follows.

Given $f, g \in k\langle x, y\rangle$ such that $[f, g] = \lambda[x, y]$, where $\lambda \in k^{\times}$, does the mapping $x \mapsto f$, $y \mapsto g$ define an automorphism of $k\langle x, y\rangle$?

In the special case where f, g reduce to $0, y$ respectively on putting $x = 0$, Czerniakiewicz [71] gives an affirmative answer. The above question may be regarded as the analogue of the Jacobian problem for polynomial rings: if k has characteristic 0 and $f, g \in k[x, y]$ are such that $\partial(f, g)/\partial(x, y) = \lambda \neq 0$, does $x \mapsto f$, $y \mapsto g$ define an automorphism of $k[x, y]$? (cf. Engel [62]).

ii) The problem of deciding when an automorphism is tame leads to a more general question: when is a given family of elements free? A family of elements $U = \{u_1, \cdots, u_n\}$ in an algebra is said to be *free* if the subalgebra generated by it is free on U. Let $F = k\langle x_1, \cdots, x_d\rangle$ be the free k-algebra of rank d, then any generating set of d elements is necessarily free (Lewin [69], Cohn [69^a], but of course not every free set of d elements need generate the whole algebra. What we need is a simple test for finding whether a given family is free; so far none is known. To state the problem more precisely, let us call two families $U = \{u_1, \cdots, u_d\}$ and $V = \{v_1, \cdots, v_d\}$ of elements in a free algebra *tamely related* (relative to a given free generating set) if we can pass from U to V by a series of elementary transformations. The problem may now be formulated as follows.

Given a finite family U in a free algebra, find a family U' tamely related to U, and of such a form that it is possible to tell by inspection whether U' is free, or a generating set (or both).

For example, in the case of groups, a positive answer is provided by the Nielsen reduction process (Hall [59], Magnus et al. [66]), and the same method can be applied to obtain the answer for Lie algebras (Cohn [64^a]). For associative algebras there are only partial results (Cohn [64^a, 69^a, 71^b]). Of course a complete answer would provide us with a method of deciding when a given subalgebra is free.

iii) That subalgebras of free associative algebras need not be free is well known; here is an amusing example (Cohn [62^a, 64^a]). For any k-algebra R, let $[R,R]$ be the ideal generated by the commutators $[x,y]$; the quotient $R/[R,R]$ is the abelianization R^{ab}. Let S be the subring $k+[R,R]$ of R; then S^{ab} contains non-zero nilpotent elements, as the identity

$$c[ac,bc] = ca[c,bc] + cb[ac,c] + c^4, \qquad \text{where} \quad c = [a,b],$$

shows. Hence S cannot be free; if it were, S^{ab} would be a polynomial ring.

Lewin [74^a] proves that a subset Y of F is free whenever it is algebraically independent modulo the commutator ideal $[F,F]$.

In Cohn [71^b] it is shown that if an ideal of a free algebra F generates a proper subalgebra $B \neq k$, then B is not free. This is also a consequence of the following result. We recall that the *idealizer* of a right ideal $\mathfrak{a}$ in any ring R is the ring $I(\mathfrak{a}) = \{u \in R \mid u\mathfrak{a} \subseteq \mathfrak{a}\}$.

THEOREM 4.2 (Dicks [74]). *Let R be an integral domain and S a subring which satisfies* WA_2 *relative to some filtration. Then for any nonzero right ideal $\mathfrak{a}$ of R, if $\mathfrak{a} \subseteq S$, then $I(\mathfrak{a}) \subseteq S$.*

Thus over a given right ideal $\mathfrak{a}$, the idealizer is a lower bound of subrings with WA_2. Dicks observes that in a free algebra idealizers of principal right ideals are often free, but he shows that $xyx - 1$ has a non-free idealizer in $k\langle x,y\rangle$. Nevertheless, as he points out, this element has another weaker property (than freeness of the idealizer). Let $\mathfrak{a}$ be a right ideal in an integral domain R; then Dicks calls $\mathfrak{a}$ *tense* in R if the sequence

$$0 \to \mathfrak{a} \to I(\mathfrak{a})$$

remains exact on tensoring on the right by $\mathfrak{a}$ (over $I(\mathfrak{a})$). If $I(\mathfrak{a})$ is a free algebra, then $\mathfrak{a}$, as ideal in $I(\mathfrak{a})$, is a free left $I(\mathfrak{a})$-module and so $\mathfrak{a}$ is then tense. Thus the tenseness of $\mathfrak{a}$ may be regarded as a weak form of freeness of $I(\mathfrak{a})$; it is related to the property of being tensorial (Cohn [70^b]). No examples are known of principal right ideals in a free algebra that are not tense.

One way of proving that subalgebras are free is to use the weak algorithm: clearly any subalgebra of $k\langle X\rangle$ satisfying WA relative to the X-degree is free, and this raises the following question: given a free algebra F with subalgebra G which is also free, does there exist a filtration v on F such that both F and G satisfy WA relative to v? Such a subalgebra G is said to be *regularly embedded* in F. An example of an irregularly embedded free algebra has been found by Dicks [74]: Let $F = k\langle x,y\rangle$, $a = xyx-y$, and $G = I(aF)$, the idealizer of aF. He shows that G is free by verifying the WA relative to the y-degree filtration but for any degree function w in which $w(x) > 0$, G does not even satisfy WA_2.

iv) Let R be any k-algebra; a subalgebra S of R is said to be *pure* (Koševoi [71]) if, for any non-scalar polynomial $f \in k[t]$ in a commuting indeterminate t and any $c \in R$, $f(c) \in S$ implies $c \in S$. Any pure subalgebra of $k\langle X\rangle$ shares with $k\langle X\rangle$ the property expressed by Bergman's centralizer theorem (Bergman [67, 69], Cohn [71^b]): the centralizer of any non-scalar element in $k\langle X\rangle$ is a polynomial ring in one variable. For a non-trivial example of a pure subalgebra (due to Koševoi) see Cohn [71^b], Ex. 6.8.8. Any intersection of pure subalgebras is pure, so we may speak of the pure subalgebra generated by a given subset. Now Dicks [74] proves that in a free algebra the pure subalgebra generated by a non-zero principal right ideal $\mathfrak{a}$ is the idealizer of $\mathfrak{a}$; more generally this holds for any 2-fir over k which remains a 2-fir under field extension of k (more precisely, for any conservative 2-fir, cf. Cohn [71^b]).

v) In Cohn [62^a] a number of conjectures for free algebras were stated, which have analogues for groups (where they are theorems, cf. Magnus et al. [66]). Below we recall those that are still open and add some others. We shall say that $f \in k\langle x_1,\cdots,x_d\rangle$ *involves* x_1 if $f \notin k\langle x_2,\cdots,x_d\rangle$, and we write (f) for the ideal generated by f. An element of $k\langle X\rangle$ is said to be *primitive* if it forms part of a free generating set of $k\langle X\rangle$; it is probably unnecessary to warn the reader against confusing this notion with the primitivity defined in Section 3; the latter will not occur again.

CONJECTURE 1. *Freiheitssatz. Let f be an element of $F = k\langle x_1,\cdots,x_d\rangle$ which involves x_1; then $(f) \cap k\langle x_2,\cdots,x_d\rangle = 0$, hence the images of $x_2,\cdots,x_d$ in $F/(f)$ generate a free subalgebra.*

The analogue for groups was proved by Magnus (cf. Magnus et al. [66]) in 1930.

CONJECTURE 2. *A k-algebra on* $x_1, \cdots, x_d$ *with a single defining relation* $f = 0$ *is free if and only if* f *is primitive in* $k\langle x_1, \cdots, x_d \rangle$.

The analogue for groups is due to J. H. C. Whitehead [36]. A proof of Conjecture 1 would have the following conjecture as a consequence.

CONJECTURE 3. *Little Freiheitssatz. Let* f *be an element of* F *which is not a unit; then* $(f) \neq F$.

PROBLEM 4. *The word problem for rings with a single defining relation. Find an algorithm for deciding whether a given element of* $k\langle X \rangle$ *lies in* (f). *(Here it is assumed that k has a soluble word problem.)*

CONJECTURE 5. *Identity theorem. If* $f \in F = k\langle X \rangle$ *has a scalar eigenring (that is,* $I(fF) = k + fF$*), then* $(f)/(f)^2$ *is freely generated by* $f + (f)^2$ *as right* $T^{op} \otimes T$*-module, where* $T = F/(f)$.

The Freiheitssatz was proved in the special case of a homogeneous defining relation such that $F/(f)$ is an integral domain, by Lewin-Lewin [68], and Lewin [74^a] proves the identity theorem when $T^{op} \otimes T$ is an integral domain. Dicks [74] obtains partial results which in particular enable him to prove conjectures 1, 3, 4, 5 in the case where f is a Lie element. He also (in Dicks [72, 74] obtains another proof of the Freiheitssatz for Lie algebras (cf. Širšov [62], Labute, unpubl.).

An interesting property of zero-divisors was observed by G. M. Bergman (cf. Lewin [74^a]). It is the case $r = 1$ of the following result: the proof for $r = 1$ still applies in the general case.

THEOREM 4.3. *Let* F *be a free algebra,* $\mathfrak{a}$ *an ideal in* F *and* $n \geqq 1$. *Then a family* $c_1, \cdots, c_r$ *of elements of* F *maps to a right linearly independent family in* $F/\mathfrak{a}$ *if and only if it maps to a right linearly independent family in* $F/\mathfrak{a}^n$.

Another open problem of some interest is the determination of the projective objects in the category of associative algebras. It is not hard to see that such an object is a subalgebra B of a free algebra F which is complemented by an ideal $\mathfrak{a}$:

$$F = \mathfrak{a} \oplus B.$$

It is plausible that such an algebra B must be free; when B has at most two generators, this can be proved as follows. If the generators of B do not commute, they generate B freely Cohn ([71^b], Prop. 6.8.2, Cor.). So we may assume that B is generated by two commuting elements u, v, and by Bergman's centralizer theorem, B is contained in a subalgebra which is a polynomial ring $k[c]$. Now there is a surjective homomorphism $F \to B$ obtained by mapping $\mathfrak{a}$ to 0. Let $c \mapsto b$ under this homomorphism, then B is generated by b; either $b \notin k$, then B is free on b, or $b \in k$, then $B = k$, that is, B is free on the empty set.

For more than two generators the problem is still open.

5. Power series and links with language theory

i) Besides the ascending filtration, by degree, the free algebra $k\langle X\rangle$ also has a descending filtration, using the *order*, that is, the least degree of terms occurring. This leads to the *inverse* weak algorithm; we shall not repeat the definitions (Cohn [62^b, 71^b], Bergman, [67]) but merely observe that it carries over to the power series completion $k\langle\langle X\rangle\rangle$ and can be used to derive certain unique factorization properties of this ring (it is a "rigid" UFD). A general method of obtaining rings with inverse weak algorithm is described in Cohn [70^b]:

THEOREM 5.1. *If R is a ring with a two-sided ideal $\mathfrak{a}$ such that $R/\mathfrak{a}$ is a skew field, $\bigcap \mathfrak{a}^n = 0$ and $\mathfrak{a}$ is free as right R-module, then R satisfies the inverse weak algorithm relative to the $\mathfrak{a}$-adic filtration.*

In particular, the conditions are always satisfied if R is a fir and $\mathfrak{a}$ an ideal such that $R/\mathfrak{a}$ is a field. The ring $k\langle X\rangle$ is a case in point; however, it should be noted that when X has more than one element, the power series ring $k\langle\langle X\rangle\rangle$ is no longer a fir, but merely a semifir (Cohn [71^b]).

Brumer [68] has shown that for any pseudo-compact commutative ring Λ, $\text{gl.dim}\,\Lambda\langle\langle X\rangle\rangle = \text{gl.dim}\,\Lambda + 1$, and comparing this result with Theorem 2.4 naturally leads one to ask whether the latter has an extension to "continuous free products".

The power series ring is useful for studying a number of questions; for example, automorphisms take a very simple form, and in fact Makar-Limanov's proof of Theorem 4.2 operates in the power series ring. The notion has found particular application in the study of algebraic language theory, and this in turn promises to throw light on certain subrings of the power series ring. We pause briefly to describe the connexion, referring to Chomsky-Schützenberger [63], Ginsburg [66], Eilenberg [74] or Cohn [75^b] for details.

ii) Let X be a finite set and X^* the free monoid on X. By a *language* on the *alphabet* X one understands a subset of X^*. To be interesting, this concept has to be restricted somewhat, and one does this by considering languages that can be generated by certain sets of rules, called *grammars*, and then classifies languages by the complexity of the grammars capable of generating them. The type of grammar (and the language produced) that has been most studied mathematically is the *context-free* grammar, and the subclass of *regular* (or *finite-state*) grammars. Given a language, such as $\{1, x, x^2, \cdots\} = \{x^n\}$, to determine whether or not it is regular we must find a regular grammar which generates it. In fact $\{x^n\}$ is regular; the language $\{x^n y^n\}$ is context-free but not regular. It is clear from this description that the task of determining whether a given language is context-free or regular is far from easy: the language may be given by a grammar that is not context-free; what we have to do is to find whether a context-free grammar exists generating the same language. The general problem of deciding whether a given language is context-free is unsolvable; this adds interest to the following alternative description of languages in terms of power series.

With each language L in an alphabet X we associate an element f of $\mathbf{Z}\langle\langle X\rangle\rangle$, namely

$$f = \Sigma\, a_w w, \tag{1}$$

where w runs over X^* and a_w is the characteristic function of L. Sometimes one may use a "weighted" sum, in which a_w represents the number of ways in which the word w can be derived from the rules of a particular grammar generating L (this associates (1) with a grammar rather than the language, unless we assign weights to each word in L, cf. Shamir [67]).

Let Λ be any commutative ring. An element of $\Lambda\langle\langle x\rangle\rangle$ is called *rational* if it can be obtained as a component of the solution of a matrix equation

$$Au - a = 0, \tag{2}$$

where a is a column over $\Lambda\langle X\rangle$ and A is a matrix over $\Lambda\langle X\rangle$ which is invertible over $\Lambda\langle\langle X\rangle\rangle$. We observe that A is invertible precisely if its image $\varepsilon(A)$ under the augmentation ε is invertible over Λ. Multiplying A by $\varepsilon(A)^{-1}$, we may take it in the form $A = I - B$, where B has entries with zero constant terms. Thus (2) may be rewritten as

$$u = Bu + b, \tag{3}$$

or in components, $u_i = \Sigma\, b_{ij} u_j + b_i$ (b_i, $b_{ij} \in \Lambda\langle X\rangle$, $\varepsilon(b_{ij}) = 0$).

Suppose now that we have a system of equations

$$u_i = \phi_i(u, x) \tag{4}$$

where ϕ_i is a polynomial in the u's and x's without constant terms or linear terms in the u's. Then we can solve for u_i by successive substitution and obtain a unique solution of (4) in $\Lambda\langle\langle X\rangle\rangle$; any component of such a solution is said to be *algebraic*. Clearly (3) is a special case of (4), thus every rational element is algebraic; moreover, it is not hard to verify that the rational and algebraic power series each form a subalgebra of $\Lambda\langle\langle X\rangle\rangle$:

$$\Lambda\langle X\rangle \subseteq \Lambda^{\mathrm{rat}}\langle X\rangle \subseteq \Lambda^{\mathrm{alg}}\langle X\rangle \subseteq \Lambda\langle\langle X\rangle\rangle.$$

More generally, these definitions still apply when the coefficients Λ form a *semiring* (which differs from a ring only in lacking subtraction), e.g. the semiring N of non-negative integers. Now the basic link with language theory is the following theorem.

THEOREM 5.2. (i) *A language is regular if and only if it is the support of a rational power series* over N (Kleene [56]);

(ii) *a language is context-free if and only if it is the support of an algebraic power series* over N (Schützenberger [62]).

For example, $\Sigma x^m y^n$ is rational, for it has the form $(1 - x)^{-1} (1 - y)^{-1}$. The series $v = \Sigma x^n y^n$ is algebraic, for it can be obtained by solving the equation

$$v = xvy + 1.$$

However, it is not rational; this is not obvious but it follows from

THEOREM 5.3. (cf. Arbib [69], Fliess [71]). *Let*

$$f = \Sigma\, a(r)r \in k\langle\langle X\rangle\rangle$$

be rational but not in $k\langle X\rangle$. *Then there exist* $u, v, w \in X^*$, $w \neq 1$, *such that* $\Sigma a(uw^n v)\, t^n$ (*where* t *is an indeterminate*) *is not a polynomial in* t.

Another criterion for rationality, this time sufficient as well as necessary, can be given as follows. With each series $f = \Sigma a_w w$ we associate the infinite matrix (a_{uv}) whose rows and columns are indexed by X^*. This is the *Hankel matrix* of f; its rank can be defined as the maximum of the ranks of the finite submatrices. Now f is rational if and only if its Hankel matrix has finite rank (Schützenberger [61], Fliess[74^a]). From this criterion it is clear that $\Sigma x^n y^n$ is not rational. This example also shows that the test does not apply to power series in several commuting indeterminates.

iii) In the free algebra, and more generally in the free power series ring, we can equate the cofactors of any monomial, that is, if $\Sigma u_i f_i = \Sigma u_i g_i$, where the u_i are elements of X^*, none of which is a right multiple of any other, then $f_i = g_i$. The operation $\Sigma u_i f_i \mapsto f_v$ (for each v) is a linear mapping called *right transduction* with respect to the monomial u_v. It is an important fact that any transduction maps each of the rings $k^{\mathrm{rat}}\langle X\rangle$, $k^{\mathrm{alg}}\langle X\rangle$ into itself (cf. Nivat [68], Fliess [72]). With the help of this fact one proves easily that $k\langle\langle X\rangle\rangle$ is faithfully flat over $k^{\mathrm{rat}}\langle X\rangle$, and it follows that $k^{\mathrm{rat}}\langle X\rangle$ is a semifir, 1-inert in $k\langle\langle X\rangle\rangle$ (that is, every factorization of an element of $k^{\mathrm{rat}}\langle X\rangle$ in $k\langle\langle X\rangle\rangle$ can be reduced to a factorization in $k^{\mathrm{rat}}\langle X\rangle$ on multiplying by a unit). It is not known whether $k^{\mathrm{rat}}\langle X\rangle$ is a fir; an affirmative answer would provide an example of a fir which is a local ring, but not principal (Cohn [71^b], Problem 3.4.7).

The same argument applies to show that $k^{\mathrm{alg}}\langle X\rangle$ is a semifir, 1-inert in $k\langle\langle X\rangle\rangle$, but $k^{\mathrm{alg}}\langle X\rangle$ is known not to be a fir; the argument of Cohn [71^b] , Ex.5.8.6. shows that this ring has countably generated right ideals that are not free.

iv) An important problem is the study of the intersection of two languages. It is easily seen that the intersection of two regular languages is regular, but the intersection of two context-free languages need not be context-free, for example, $\{x^m y^m z^n\} \cap \{x^m y^n z^n\} = \{x^n y^n z^n\}$ and the latter can be shown to be not context-free. However, one has the important intersection theorem (cf., e. g., Ginsburg [66]).

THEOREM 5.4. *The intersection of a context-free and a regular language is context-free.*

In the framework of power series rings, intersections may be studied by means of the Hadamard product. Given two power series $f = \Sigma a_w w$, $g = \Sigma b_w w$, their *Hadamard product* is defined as

$$f \circ g = \Sigma a_w b_w w.$$

In the theory of complex functions this is used in the study of singularities; Jungen [31] proved that the Hadamard product of a rational and an algebraic power series in z is an algebraic power series. A number of attempts were made to extend this result to several variables, using other more complicated products, until Schützenberger [62] proved that Jungen's result extends to several *non-commuting* variables. Of course this provides another proof of Theorem 5.4. For a study of other types of product see Fliess [74^b].

6. Derivations and homology

i) Let R, A, B be rings and $\alpha\colon R \to A$, $\beta\colon R \to B$ homomorphisms. A mapping δ from R to an (A, B)-bimodule M is called an (α, β)-*derivation* if

$$(x + y)^\delta = x^\delta + y^\delta, \qquad (xy)^\delta = x^\alpha y^\delta + x^\delta y^\beta.$$

These conditions are equivalent to saying that $x \mapsto \begin{pmatrix} x^\alpha & x^\delta \\ 0 & x^\beta \end{pmatrix}$ is a homomorphism from R to $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. This definition shows that for $R = k\langle X\rangle$ and given α, β, any mapping $\delta\colon X \to M$ may be extended to a unique (α, β)-derivation over k (that is, mapping k to 0). Important special cases are obtained by taking $A = R$, $\alpha = 1$ or $B = R$, $\beta = 1$ or both. When $\alpha = \beta = 1$, we speak of *derivations*; this is the case considered below, where, moreover, $M = R$.

Little is known about derivations in free algebras; for example, one may ask whether the kernel of a derivation is always free. For some partial results see Jooste [71]. Derivations may be used to decompose a free algebra as follows. If $X = \{x_1, \cdots, x_d\}$, then there is a unique derivation into F such that $x_i \mapsto \delta_{i1}$; it is denoted by $\partial/\partial x_1$ and any derivation of this form, that is, with respect to a primitive element of F, is called a *primitive* derivation. As is well known from partial differentiation, primitive derivation with respect to x depends not merely on x, but also on the free set of which x is a part. If d is a primitive derivation of F, with respect to y, then ker d is free on $z_1, \cdots, z_r$ say, and y, $z_1, \cdots, z_r$ is a free generating set of F. Questions arising naturally in this context are:

(i) If a derivation δ assumes the value 1 on a primitive element, is δ necessarily primitive?

(ii) If a derivation δ assumes the value 1 on an element x, is x necessarily primitive?

Derivations arise naturally as "infinitesimal automorphisms", and if δ is a derivation, we expect $\exp \delta = \Sigma \delta^n/n!$ to be an automorphism. For a free algebra $\exp \delta$ is not defined, but it does exist in a power series ring, if the ground field is of characteristic 0. Thus we consider $G = k\langle\langle X\rangle\rangle$ as a topological ring; it is the completion of $k\langle X\rangle$ with respect to the topology induced by the filtration obtained from the order-function $o(f)$. A homomorphism or derivation is continuous if and only if it is o-increasing. Returning to general (α, β)-derivations for a moment, we see that if α, β are continuous homomorphisms from G to topological rings A, B respectively, then any mapping $X \to M$ into a topological (A, B)-bimodule can be extended to a continuous (α, β)-derivation δ of G. Here δ is unique, but if we drop continuity, this is no longer so; for example, the field of formal Laurent series $k((x))$ has many derivations that vanish on $k(x)$, but all except the zero derivation are discontinuous (Jooste [71]; this corrects an oversight in Bourbaki [50]).

Let δ be a continuous derivation on $k\langle\langle X\rangle\rangle$ (associated with the identity automorphisms), where char $k = 0$; then exp δ is defined and is easily seen to be an automorphism; the fixed ring of this automorphism is just the kernel of δ. Jooste [71] has shown that the kernel of any derivation of $k\langle\langle X\rangle\rangle$ (continuous or not) is a local ring and a semifir over which $k\langle\langle X\rangle\rangle$ is flat. Whether the kernel is in fact a power series ring is not known, though this is established in important cases by Jooste [71].

ii) Let F be a free group, R a normal subgroup, and R' the commutator subgroup of R; then Magnus [39] has shown that F/R' is faithfully represented in the triangular matrix group $\begin{pmatrix} F/R & \Omega \\ 0 & 1 \end{pmatrix}$, where Ω is a free F/R-module. This result has the following analogue for rings (Lewin [74[a]]): Let $F = k\langle X\rangle$, take ideals $\mathfrak{a}, \mathfrak{b}$ in F and put $A = F/\mathfrak{a}$, $B = F/\mathfrak{b}$; then $F/\mathfrak{ab}$ is faithfully represented in the matrix ring $\begin{pmatrix} A & \Omega \\ 0 & B \end{pmatrix}$, where Ω is an (A, B)-bimodule (in fact the universal module for (α, β)-derivations, where $\alpha\colon F \to A$, $\beta\colon F \to B$ are the natural mappings). Bergman-Dicks [74] have generalized this result to quite arbitrary rings by determining the kernel of the mapping

$$R \to \begin{pmatrix} A & \Omega \\ 0 & B \end{pmatrix}$$

for given surjective α, β by homological methods.

iii) We now turn to a homological characterization of free algebras. Let k be a commutative field and R any augmented k-algebra, with augmentation ideal $\mathfrak{a}_0$. Then Fröhlich [63] has established the following analogue of Baer's formula for $H_2(G)$. Given a presentation $R = F/\mathfrak{r}$, if $\mathfrak{a}$ is the augmentation ideal of F, then

$$\mathrm{Tor}_2^R(k,k) = (\mathfrak{a}^2 \cap \mathfrak{r})/(\mathfrak{r}\mathfrak{a} + \mathfrak{a}\mathfrak{r}).$$

Baer's formula (actually already described by Hopf) is as follows. Let $G = F/N$, where F is a free group, then

$$H_2(G,Z) = (F' \cap N)/(F,N),$$

where (F,N) is the subgroup generated by all commutators $(a,b) = a^{-1}b^{-1}ab$, $a \in F$, $b \in N$. The analogy becomes even closer if we take R to be the group algebra $k[G]$, then $H_2(G,Z) = \mathrm{Tor}_2^R(k,k)$.

Suppose that R is generated as k-algebra by a basis X of $\mathfrak{a}_0$ (mod $\mathfrak{a}_0^2$) and assume further that $\mathrm{Tor}_2^R(k,k) = 0$. Then $\mathfrak{r} \subseteq \mathfrak{a}^2$ and $\mathfrak{r} \cap \mathfrak{a}^2 = \mathfrak{r}\mathfrak{a} + \mathfrak{a}\mathfrak{r}$, hence $\mathfrak{r} = \mathfrak{r}\mathfrak{a} + \mathfrak{a}\mathfrak{r}$. Assuming $\mathfrak{r} \subseteq \mathfrak{a}^{n-1}$, we find from this formula that $\mathfrak{r} \subseteq \mathfrak{a}^n$, hence by induction, $\mathfrak{r} \subseteq \bigcap \mathfrak{a}^n = 0$, and so R is in fact free on X. This proves the following theorem.

THEOREM 6.1 (Knus ([68]). *Let R be an augmented k-algebra with augmentation ideal $\mathfrak{a}_0$ and assume that R is generated as k-algebra by a basis X of $\mathfrak{a}_0$* (mod $\mathfrak{a}_0^2$). *If moreover,* $\mathrm{Tor}_2^R(k,k) = 0$, *then R is free on X.*

It is easy to see that the condition on X cannot be omitted (cf. also Moran [73]).

iv) We conclude with a result on K-theory. A number of special results have suggested that free algebras behave in some ways like polynomial rings in one variable. The most general result to date was proved by Gersten [74] using Quillen's formulation of K-theory.

THEOREM 6.2 (Gersten [74]). *Let Λ be a right Noetherian ring of finite globbal dimension, then*

$$K_n(\Lambda\langle X\rangle) \cong K_n(\Lambda) \qquad (n = 0,1,2,\cdots).$$

This subsumes a large number of results obtained previously; for example $n = 1$ (Gersten [65]), $n = 2$ and Λ a field (J. Silvester [73], Cohn [72]), $n = 2$, $X = \{x\}$ and Λ a skew field (K. Dennis [70]).

7. The gocha

In finite-dimensional vector space theory the dimension is a useful invariant, and this remains true for free modules over a ring R, at least when R has invariant basis number. To extend the method to the infinite-dimensional case we need to distinguish the basis elements. A natural way of doing this is as follows.

Let R be a filtered ring and V a positively filtered free R-module, that is, a direct sum of copies of R, where in each copy the degree differs by a constant from the degree in R. If λ_n is the number of elements of degree n in a basis of V, assumed finite for each n, we can write down the formal power series in t:

$$\gamma(V: R) = \Sigma\lambda_n t^n.$$

This is called the *gocha* of V(after Golod-Šafarevič [64]). We note that V is also free as k-space and we can therefore also define a gocha relative to k:

$$\gamma(V) = \gamma(V: k) = \Sigma\alpha_n t^n,$$

where $\alpha_n = \dim_k V_n/V_{n-1}$, again assumed finite. The gocha was used in Cohn [69^a] to give a brief derivation of some results on free algebras due to Lewin [69]. We take up the subject again here to give an even briefer proof suggested by G. M. Bergman.

Let $\mathfrak{a}$ be a right ideal in R; then the exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

splits as a sequence of filtered k-spaces, hence $\gamma(R) = \gamma(\mathfrak{a}) + \gamma(R/\mathfrak{a})$, or

$$\gamma(R/\mathfrak{a}) = \gamma(R) - \gamma(\mathfrak{a}). \tag{1}$$

Now assume that R has weak algorithm and $k = R_0 = \{x \in R \mid v(x) \leqq 0\}$. If $\mathfrak{a}$ is filtered-free and B is a basis for $\mathfrak{a}$ as right R-module such that no element of B is right v-dependent on the others (that is, B is a "weak v-basis", in the terminology of Bergman [67] and Cohn [71^b]), then B is in fact right v-independent. Let β_n be the number of elements of degree n in B, so that $\gamma(\mathfrak{a}:R) = \Sigma\beta_n t^n$, then for any $u \in B$ of degree n, $\gamma(uR) = t^n\gamma(R)$, and $\mathfrak{a} = \Sigma u_i R$, where u_i runs over B, hence $\gamma(\mathfrak{a}) = \Sigma\beta_n t^n\gamma(R)$, that is,

$$\gamma(\mathfrak{a}) = \gamma(\mathfrak{a}: R)\gamma(R). \tag{2}$$

Since R has weak algorithm, we can find a right v-independent generating set X of R. If $\mathfrak{m}$ is the right ideal generated by X, then $\gamma(\mathfrak{m}: R) = \Sigma\xi_n t^n$, where ξ_n is the number of elements of degree n in X. Clearly $R = k \oplus \mathfrak{m}$, hence $\gamma(R) = 1 + \gamma(\mathfrak{m})$, and so by (2), $\gamma(R) = 1 + \gamma(\mathfrak{m}: R)\gamma(R)$, which yields, after a little rearrangement,

$$\gamma(R) = (1 - \gamma(\mathfrak{m}: R))^{-1}. \tag{3}$$

For example, if $R = k\langle x_1, \cdots, x_d\rangle$, $\deg x_i = 1$, then $\gamma(\mathfrak{m}: R) = dt$, hence $\gamma(R) = (1 - dt)^{-1}$, so in this case (2) takes the form $\gamma(\mathfrak{a}: R) = (1 - dt)\gamma(\mathfrak{a})$ (Cohn [69^a], [71^b]).

For any short exact sequence of filtered free R-modules

$$0 \to V' \to V \to V'' \to 0$$

it is clear that $\gamma(V: R) = \gamma(V': R) + \gamma(V'': R)$, and this fact has been exploited by Dicks [74] to define a gocha for any filtered R-module M with a finite resolution, or more generally, a resolution which converges in an appropriate sense, in such a way that (2) (with M for $\mathfrak{a}$) holds for such modules.

In a graded ring R with weak algorithm, let $\mathfrak{a}$ be a proper homogeneous right ideal and write $\mathfrak{b} = R\mathfrak{a}$, $Q = R/\mathfrak{b}$, $I = I(\mathfrak{a})$, $E = I/\mathfrak{a}$ (the eigenring of $\mathfrak{a}$); then if I satisfies the weak algorithm (relative to the induced filtration), Dicks [74] obtains the formula

$$\gamma(Q) = \gamma(E)\,[\gamma(E)\gamma(R)^{-1} + \gamma(\mathfrak{a}: R)]^{-1}.$$

In particular, taking R to be a free k-algebra of rank d and $\mathfrak{a} = aR$, where a is a homogeneous Lie element, he finds that $E = k$ and thus rederives the formula (cf. Labute [67]):

$$\gamma(R/RaR) = (1 - dt + t^{v(a)})^{-1}.$$

Within this framework, Dicks also obtains another proof of the Golod-Šafarevič theorem and the refinements by Vinberg and Bruck (cf. Fischer-Struik [68]).

8. Homomorphisms into skew fields

i) In commutative ring theory, localization is an important tool. Briefly, from any commutative ring R ($\neq 0$) there is a homomorphism f into a field k with a given ideal $\mathfrak{p}$ of R as kernel if and only if $\mathfrak{p}$ is a prime ideal in R; moreover, assuming k to be the subfield generated by im f, we can obtain k as the residue class field of the local ring $R_{\mathfrak{p}}$ by its maximal ideal. The situation is summed up in the commutative diagram shown in Fig. 2.

Here n, n' are the natural mappings to residue class rings, l is the localization

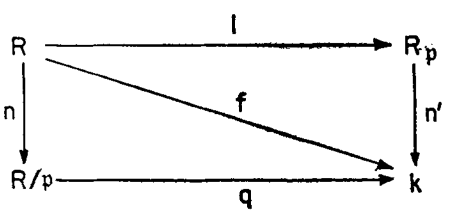

Fig. 2

(forming fractions with denominator outside $\mathfrak{p}$), and q is the embedding of the integral domain $R/\mathfrak{p}$ in its field of fractions.

Of the two ways of analysing f given in this diagram, the one in the upper right-hand corner:

$$f = ln'$$

turns out to be easier to generalize to the non-commutative case. Let us briefly sketch the general theory before seeing how it applies to free algebras.

Let R be any ring and $f: R \to K$ an epimorphism to a field, that is, a homomorphism such that K is the field generated by $\operatorname{im} f$. Such a K is called an *epic R-field*. The ideal $\mathfrak{a} = \ker f$ is not in general enough to determine K (see below), but K is completely determined by its *singular kernel*, that is, the set $\mathscr{P}$ of all square matrices over R which map to singular matrices over K. Such a set $\mathscr{P}$ is called a *prime matrix ideal* and one can write down a simple set of axioms characterizing such sets (reminiscent of the axioms for prime ideals, cf. Cohn [71^b], Chapt. 7). Given an epic R-field K, with prime matrix ideal $\mathscr{P}$ as singular kernel, let $\mathscr{P}'$ be the set of all square matrices over R *not* in $\mathscr{P}$ and denote by $R_{\mathscr{P}}$ the universal $\mathscr{P}'$-inverting ring, that is, the ring obtained by adjoining elements that constitute inverses for the matrices of $\mathscr{P}'$. (For each $n \times n$ matrix A in $\mathscr{P}'$ we adjoin n^2 indeterminates a'_{ij} forming a matrix A', and add the defining relations, in matrix form: $AA' = A'A = I$.) Then $R_{\mathscr{P}}$ can be shown to be a local ring, that is, the non-units form an ideal, and its residue class field is isomorphic to K.

Given prime matrix ideals $\mathscr{P}_1$, $\mathscr{P}_2$ with corresponding epic R-fields K_1, K_2, one defines a *specialization* from K_1 to K_2 to be a homomorphism from a local subring of K_1 containing the image of R onto K_2. It is not hard to show that a specialization $K_1 \to K_2$ exists precisely when $\mathscr{P}_1 \subseteq \mathscr{P}_2$. Thus if there is a least prime $\mathscr{P}$, it corresponds to a *universal R-field U*; by definition this is an epic R-field with a unique specialization (up to R-equivalence) to any other epic R-field. If $\mathscr{P}$ is a least prime and contains no 1×1 matrices other than 0, then R is embedded in U and U is called the *universal field of fractions* of R. In the commutative case R has a universal field of fractions if and only if R is an integral domain, that is, 0 is a prime ideal, but in the general case the situation is more complicated, because certain non-zero matrices will always map to singular matrices.

Let A be an $n \times n$ matrix over a ring R. If $A = PQ$, where P is $n \times r$, Q is

$r \times n$ and $r < n$, then A is said to be *non-full*, otherwise A is *full*. For example, a 1×1 matrix is full if and only if it is non-zero; a typical non-full 2×2 matrix has the form

$$\begin{pmatrix} a \\ b \end{pmatrix} (c \;\; d) = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}.$$

Clearly any prime matrix ideal contains all non-full matrices. If the set $\mathcal{N}$ of all non-full matrices is itself a prime matrix ideal, then the corresponding R-field U is a universal field of fractions. Now we have the following result (Cohn [71^b], p. 283).

THEOREM 8.1. *In any semifir R, the complement of the set Φ of all full matrices is a prime matrix ideal. Hence R has a universal field of fractions U, over which every full matrix of R has an inverse; U is obtained as the universal Φ-inverting ring.*

A full development of this approach leads to a set of necessary and sufficient conditions for a general ring to be embeddable in a skew field (cf Cohn [71^b], Chapt. 7; for an interesting set of necessary—but not sufficient—conditions, see Klein [69]). As predicted by general theory (Cohn [65], Mal'cev [73]) these conditions take the form of an infinite set of quasi-identities. Moreover, the existence, for any n, of an n-fir not embeddable in a field (Cohn [69^b, 74^b]) shows that the conditions cannot be replaced by a finite set of conditions; this answers a question raised by Mal'cev [73].

The embedding $R \to U$ described in Theorem 8.1 is particularly useful because any automorphism of R can be extended to an automorphism of U. More generally, let us call a ring homomorphism *honest* if it keeps full matrices full, then we can easily deduce the following.

COROLLARY. *Any honest homomorphism between semifirs extends to a homomorphism between their universal fields of fractions; likewise any derivation of a semifir extends to a derivation of its universal field of fractions.*

ii) Since free algebras are semifirs, Theorem 8.1 and its Corollary can be applied in this case. In the construction the elements of the universal field of fractions are obtained explicitly as the components of the solutions of matrix equations. We cannot expect to get a normal form, since this does not even exist for general integral domains in the commutative case, but there is a procedure for deciding when two expressions represent the same element, that is, the word problem has a solution (cf. Cohn [73^c]). Of course this is far from true for general skew fields (cf. Macintyre [73]).

Let $k(x_1, \cdots, x_d)$ be a commutative pure transcendental extension. It is well known that in the case $d = 1$, every subextension is again pure; this is Lüroth's theorem and it does not (without restricting k) extend to the case $d > 1$. This suggests the following conjecture (cf. Cohn [73^d]).

CONJECTURE 6. *Any finitely generated subfield of the universal field of fractions U of $k\langle X\rangle$ is free, that is it is again the universal field of fractions of a free algebra.*

A related conjecture is the following.

CONJECTURE 7. *Let U be the universal field of fractions of a free algebra. Then any family of r elements of U is either a free generating set, or is contained in a subfield of U on fewer than r generators.*

If this were true, it would follow (by an easy induction) that every non-field-free family of r elements is contained in a free subalgebra of lower rank than r.

The corresponding conjecture for algebras would imply that projectives are free (cf. to Section 4), but it is not likely to be true, as the following remark of Bergman's shows. In $R = k\langle x, y, z\rangle$ take $u = xyxz + xy$, $u' = zxyx + yx$, $v = xyx$; then $uv = vu'$, but there does not seem to be a 2-generator subalgebra of R containing u, u', v. However, in the free field on x, y, z, the subfield generated by u, u', v is contained in the subfield generated by v and $z + x^{-1}$.

Here is a simple application of Theorem 8.1 and its Corollary which has not appeared in print before. Let p be a prime number and k a commutative field. If char $k \neq p$, we also assume that k contains a primitive pth root of 1, ω say; otherwise (when char $k = p$) we set $\omega = 1$. Form the free algebra $F = k\langle x_{hij}\rangle$ $(h, i, j = 1, 2, \cdots)$ and let E be its universal field of fractions. On F we have an endomorphism $S: x_{hij} \mapsto \omega^{-i}\, x_{hij+1}$, which is clearly honest and so extends to an endomorphism of E, again denoted by S. Now the mapping $D: x_{hij} \mapsto x_{hi+1j}$ extends to a $(1, S)$-derivation D of F and hence of E (because E is constructed as localization of a free algebra, cf. Th. 8.1). It is easily checked that $SD = \omega DS$. We now form $L = E(t; S, D)$, the field of fractions of the skew polynomial ring $E[t; S, D]$, and denote by K the subfield of L generated by E and t^p. Then L/K is a binomial extension of right degree p (cf. Cohn [66]: this means that $1, t, \cdots, t^{p-1}$ form a right K-basis for L). But the left degree is infinite, for the elements x_{h11} can be shown to be left linearly independent over L^S. By iterating the construction we obtain a skew field extension of arbitrary finite right degree and infinite left degree. Further, by taking the index set over which h in x_{hij} ranges to be of suitable size we obtain extensions of arbitrary finite right degree and arbitrary infinite left degree.

This provides a simple example answering a problem raised by E. Artin (for other examples see Cohn [61^c, 66]). Whether skew field extensions with different left and right degrees, *both finite*, exist is still open.

C. M. Ringel has asked (in conversation) whether extensions L/K with different left and right degrees can be constructed in which $L \cong K$; in the above construction this is indeed the case when k has characteristic p, so that $\omega = 1$.

The same method can be used to construct a simple principal right ideal domain which is not left Noetherian (cf. Cozzens [72]).

These constructions depend essentially on the property of the universal field of fractions expressed in the corollary to Theorem 8.1. For other applications, to the free product of skew fields, see Cohn [71^a], [73^d].

iii. Several other methods of embedding free algebras in fields are well known:

(a) The Jategaonkar-Koševoi method is in some ways the simplest. It is based on the fact that any right Ore domain has a field of fractions and any integral domain which is not left Ore contains a free algebra of rank 2 (and hence a free algebra of countable rank). Thus any right but not left Ore domain over a commutative field k provides an embedding of $k\langle x, y\rangle$ into a field (Jategaonkar [69], Koševoi [70]). The drawback is that it is in general not possible to extend honest endomorphisms (or even automorphisms) of $k\langle x, y\rangle$ to the field of fractions.

(b) The Mal'cev-Neumann method depends on the fact that for any totally ordered group G, the set $k((G))$ of "power series" over k with well-ordered support forms a field containing the group algebra $k[G]$. Since the free group can be ordered, this provides an embedding of the free algebra in a field (Mal'cev [48], Neumann [49], Cohn [65]). Lewin [74^b] shows that the subfield of $k((G))$ generated by $k[G]$ is in fact the universal field of fractions of the latter, in other words, the inclusion $k[G] \to k((G))$ is honest.

Clearly any order-preserving automorphism of G can be extended to an automorphism of $k((G))$. Moreover, if the embedding $k[G] \to k((G))$ is honest (for example, when G is free), any homomorphism of ordered groups $f: G \to H$

induces a specialization from $k((G))$ to $k((H))$. Consider the subset L of $k((G))$ consisting of all series in which the sum over any coset mod ker f is finite. It is clear how to define $f_1: L \to k((H))$ so as to extend f; if we now localize at the set of matrices inverted over $k((H))$ we get a local ring containing L, on which f_1 can be defined, with $k((H))$ as residue class field.

(c) A third method depends on the following theorem.

THEOREM 8.2 (Cohn [61^b]). *Let R be a filtered ring such that the associated graded ring satisfies the Ore condition. Then R is embeddable in a field.*

If L is a Lie algebra over a commutative field k, its universal associative envelope U is filtered by the powers L^i and the associated graded ring is a polynomial ring in $\dim_k L$ commuting indeterminates. Hence U is embeddable in a field. In particular, this applies to $k\langle X\rangle$, which may be regarded as the associative envelope of the free Lie algebra on X.

iv) The problem of embedding a ring in a skew field has a parallel in the problem of embedding a ring in a radical ring. Any ring (associative, but not necessarily with 1) is a monoid under the operation of quasi-multiplication

$$a \circ b = a + b - ab$$

obtained by simplifying the relation $1 - a \circ b = (1 - a)\ (1 - b)$, where 1 is a formal symbol. If R forms a group under this operation, it is said to be a *radical ring*; the condition is equivalent to requiring that R coincide with its Jacobson radical, in particular it cannot then have a unit-element. From this definition it is clear that radical rings form a variety in the sense of universal algebra: they are rings with a unary operation $x \mapsto x'$ satisfying $x + x' = xx' = x'x$. Thus we may speak of the *free* radical ring on a given set X. For example, the free radical ring on one generator x is the subring of the field $Q(x)$ consisting of all expressions $xf/(1+xg)$, where $f, g \in Z[x]$. For more than one generator the free radical ring is more complicated and no normal form is known, but one has the commutative diagram (by general nonsense) as shown in Fig. 3.

Here $k^0\langle X\rangle$ is the free algebra without 1 (consisting of all elements in $k\langle X\rangle$ without constant term), P is the free radical k-algebra on X, and the mapping g exists because $k^0\langle\langle X\rangle\rangle$ (the ring of power series without constant term) is clearly radical. An obvious problem is to know whether g is injective; an affirmative answer will provide a representation for the elements of P which is of interest in

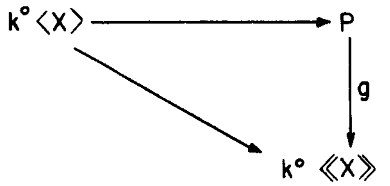

Fig. 3

the study of the group ring of the free group. The free group on a set S equipotent to X can be embedded in P^1 (the ring obtained by adjoining a unit-element to P) by mapping $s \mapsto 1 - x$, $s^{-1} \mapsto 1 - x'$. Now the above problem can be solved as follows. One has (Cohn [73^b]) the following theorem.

THEOREM 8.3. *Let R be a k-algebra which is embedded in a radical k-algebra S. If R^1 is totally inert in S^1, then the radical subalgebra of S generated by R is the free radical k-algebra on R.*

Here "totally inert" means, roughly speaking, that if a matrix over R^1 is factorized over S^1, then this factorization can be reduced to one over R^1 on multiplying by an invertible matrix (see Cohn [71^b] Ch. 1. For example, a totally inert embedding is necessarily honest. Now the inertia theorem proved in Cohn [71^b] states that $k\langle X\rangle$ is totally inert in $k\langle\langle X\rangle\rangle$ (the case of 1-inertia was first proved in Bergman [67]). It follows that the mapping g above is injective.

For another application of field products, to the construction of simple radical rings, see Cohn [71^d].

v) Localization in non-commutative rings may be carried out in several different ways. A frequently used method is the categorical one based on Gabriel's construction (Gabriel [62]). This may be called the *injective* method, and it has mainly been applied to Noetherian rings, (cf. Lambek-Michler [73], andfor an interesting attempt to obtain algebraic varieties by localizing at prime ideals, Murdoch-v. Oystaeyen [75]). We shall not be concerned with this type of localization, but rather with the "inversive" method, which consists in making certain elements or matrices invertible. For an application of this method to Noetherian rings, see Cohn [73^e]; below we outline a description of the method for semifirs, with an application to group rings of free groups.

We recall that if R is a ring and Σ a set of square matrices over R, a homomorphism $f: R \to S$ is said to be *Σ-inverting* if every matrix in Σ is mapped by f

to an invertible matrix over S. For any R and Σ there always exists a universal Σ-inverting ring R_Σ whose construction was described earlier (in i); it is a ring with a Σ-inverting homomorphism $\lambda: R \to R_\Sigma$ such that every Σ-inverting homomorphism can be factored uniquely by λ. This much is trivial, what one would like to know is: under what circumstances is λ injective?

For example if R is a $2n$-fir with left and right ACC_n and Σ consists of full matrices of order at most n, then λ is injective (Cohn [71[a]]); in particular, this shows that for an atomic 2-fir R the monoid $R^\times$ is embeddable in a group. One would imagine that atomicity is not needed here, and that more generally, λ is injective when R is a $2n$-fir and Σ consists of full matrices of order at most n.

There is a more useful result in a somewhat different direction. A set Σ of square matrices is called *factor-closed* if for any square matrices A, B of the same order, $AB \in \Sigma$ implies $A, B \in \Sigma$. Now we have

THEOREM 8.4 (Cohn [74[a]]). *Let R be a semifir and Σ a factor-closed set of full matrices over R. Then the natural mapping $\lambda: R \to R_\Sigma$ is an embedding and R_Σ is again a semifir.*

In a free algebra $k\langle X\rangle$, X is clearly factor-closed and full, hence this provides another proof that the group algebra of the free group is a semifir, for we can write it as a localization of $k\langle X\rangle$ with respect to X.

The injectivity of λ in Theorem 8.4 already follows from Theorem 8.1; what is new is the assertion that the localization is a semifir. The condition on Σ (to be factor-closed) cannot be omitted: the universal xy-inverting ring over the free algebra $k\langle x, y\rangle$ is not even an integral domain, let alone a semifir: for $y(xy)^{-1}x$ is clearly idempotent, but is neither 0 nor 1 in this ring.

A ring is said to be *weakly finite* if for any two matrices A, B of the same order, $AB = I$ implies $BA = I$. Clearly every ring R has a universal weakly finite image $w(R)$. The following consequence of Theorem 8.4 was observed by G.M. Bergman.

COROLLARY. *Let R be a semifir and Σ any set of full matrices over R, then $w(R_\Sigma)$ is a semifir, and it is isomorphic to $R_{\Sigma'}$ where Σ' is the "factor-closure" of Σ.*

One would like to have an analogue of Theorem 8.4 for firs; although the proof of Theorem 8.4, as it stands, cannot be adapted, such a result may well hold. On the other hand, examples in Cohn [74[a]] show that the analogue for n-firs (for a fixed n) is false, but Bergman has suggested the following conjecture.

CONJECTURE 8. *If R is an $f(m,n)$-fir and Σ a factor-closed set of full $n \times n$*

matrices over R, then R_Σ *is an m-fir. Here* $f(m, n)$ *is a function to be determined (perhaps* $f(m, n) = 2mn$; *this seems likely to be the correct value for* $m = n = 1$).

9. The field spectrum of a ring; equations over skew fields

i) Let R be any ring and $X = X(R)$ the set of (isomorphism classes of) epic R-fields, partially ordered by specialization. We may think of X more concretely as the set of primes (that is, prime matrix ideals) of R, partially ordered by inclusion. This way of looking at X also reassures us that it really is a set. Now each square matrix A over R determines a subset of X, called the *non-singularity support* of A:

$$D(A) = \{x \in X \mid A \notin \mathscr{P}_x\},$$

where $\mathscr{P}_x$ is the prime corresponding to x. It is easily seen that

$$D\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = D(A) \cap D(B), \qquad D(I) = X.$$

Hence the $D(A)$ admit finite intersections, and so form a base for the open sets of a topology on X. The topological space so defined is called the *field-spectrum* of R; it is analogous to the usual prime spectrum of a commutative ring, to which it reduces when R is taken to be commutative. Moreover, it satisfies the conditions for a spectral space as defined by Hochster [69] (cf. Cohn [72^b]).

For a commutative ring R, X has a least element (that is, a unique minimal element) if and only if the nilradical of R is prime, and this least element corresponds to a field embedding precisely when R is an integral domain. For a general ring there may be many primes that correspond to embeddings, for example, in the case of a free algebra of rank at least two, Fisher [74] has shown that the subset of X consisting of primes that correspond to field embeddings includes a set order-isomorphic to the set of all subsets of an infinite set.

Following the pattern of the commutative case one can, for any ring R, form a sheaf $\tilde{R}$ of local rings on X; the stalk at $x \in X$ is the local ring $R_x (= R_{\mathscr{P}_x})$, and the correspondence $R \mapsto \tilde{R}$ is a contravariant functor (see Hofmann [72] for a survey of such representations). The global sections of $\tilde{R}$ may be looked on as generalized fractions, given by different expressions in different open sets covering X. Of course each $a \in R$ defines a global section $\hat{a}$, and the mapping $a \mapsto \hat{a}$ is a homomorphism

(1) $$\gamma: R \to \Gamma(X, \tilde{R})$$

from R to the ring of global sections, sometimes called the *Gelfand morphism*.

For commutative rings (1) is always an isomorphism (Grothendieck-Dieudonné [71]), but for general rings this need not be so, for example, X may be empty. However, for the free algebra F, γ is necessarily injective. Whether γ is surjective for F is not known; this is an interesting but difficult question. In an attempt to make it more tractable we split it into two questions (both meaningful for any ring).

A matrix A over a ring R is said to be *proper* if some homomorphism of R into a field maps A to a singular matrix, otherwise A is *improper*. An invertible matrix is necessarily improper, and over a commutative ring the converse holds: if A is a non-unit, then so is det A, and there is a homomorphism to a field which maps det A to 0. But over a general ring there may be improper matrices that are non-invertible, for example, if R is a simple ring not a field, then any non-zero 1×1 matrix is improper, for any homomorphism of R into a field is necessarily an embedding, hence a zero-divisor in R cannot be inverted. Now each improper matrix A provides us with global sections: A^{-1} exists in every localization R_x and each entry of A^{-1} thus defines a global section. In general these sections will not be in im γ (unless A is invertible over R) and we shall call such sections *rational* in contrast to the *integral* sections, which lie in the image of γ. Now we can say for any ring R, that γ is surjective if and only if

(a) every global section is rational, and

(b) every rational section is integral.

For a free algebra (more generally, for any ring R such that ker γ is contained in the Jacobson radical of R), (b) holds if and only if every improper matrix is invertible, and we may state the following conjecture.

CONJECTURE 9. *Let K be a skew field and k a central subfield of K which is algebraically closed. Then over the free algebra $K_k^* k\langle X\rangle$, any improper matrix is invertible.*

We shall see later that some condition on k is necessary. For the moment we observe that when K is finite-dimensional over k, K must be commutative and a purely inseparable extension of k. In that case Conjecture 9 can be verified for 2×2 matrices, but the more general case is still open.

An affirmative answer to this conjecture would confirm (b) for free algebras and also have other interesting consequences, described below. By contrast, (a) seems of less interest; although it probably holds for free algebras, it is very unlikely to hold for all rings, though no explicit counter-example is known. Here is a candidate for a counter-example, suggested by Bergman. Let R be the k-algebra on four generators a_{ij} $(i, j = 1, 2)$ with defining relations (in matrix form) $A^2 = A$, where $A = (a_{ij})$.

In every R-field we can define $\mathrm{rk}(A)$, the rank of A, and it is clear that the subset of $X(R)$ where $\mathrm{rk}(A) \leqq n$ is closed, for each n. Likewise the set where $\mathrm{rk}(I - A) \leqq 2 - n$ is closed, that is, where $\mathrm{rk}(A) \geqq n$. It follows that $X(R)$ is the union of three open-closed sets, defined by the conditions $\mathrm{rk}(A) = 0, 1, 2$. The characteristic functions of these sets are clearly global sections of Γ, but there is no reason to believe that they are rational. If R were commutative (that is, if commutativity were imposed), these sections could be expressed in terms of determinants, but in the non-commutative case this is no longer so.

ii) In order to do non-commutative algebraic geometry one must be able to solve algebraic equations over skew fields. Thus, given a polynomial $f(x) = f(x_1, \cdots, x_d)$ with coefficients in a field K we look for a larger field L in which $f(x) = 0$ has a solution. Here the x's must not be allowed to commute with the coefficients, but in any case they will commute with certain coefficients, such as $1, -1, \cdots$ and, to be precise, we shall prescribe the subfield k of coefficients commuting with the x's. As in Section 2 we find that k must be contained in the centre of K, thus we have a field K which is a k-algebra, and we form the free K-ring in X over k:

$$F = K_k^* k\langle X \rangle.$$

By a *polynomial* in $X = \{x_1, \cdots, x_d\}$ over K/k we understand an element of this ring F, and a plausible guess might be that every non-constant polynomial has a solution in some extension field. This is false in general, as shown in the following example (obtained in the course of a discussion with S. A. Amitsur).

Let a be an element of K which is separable over k but not in k, and consider the equation

$$ax - xa = 1 \qquad \text{(Metro-equation).} \tag{5}$$

Suppose that (5) has a solution $x = \xi$ in some extension; (5) may then be written $[a, \xi] = 1$, and by a familiar argument it follows that $[f(a), \xi] = f'(a)$, where f' is the derivative, for any polynomial f with coefficients in k. Taking f to be the minimal polynomial of a over k, we obtain $f'(a) = 0$, an equation of lower degree (because a was separable). This contradiction shows that (5) cannot have a solution and it explains the necessity of the condition in the following conjecture.

CONJECTURE 10. *Let K be a field which is a k-algebra, where k is algebraically closed. Then every non-constant equation in $x_1, \cdots, x_d$ over K/k has a solution in some extension.*

We observe that Conjecture 10 is a special case (the case of 1×1 matrices) of Conjecture 9; this also explains the necessity to restrict k in Conjecture 9.

Even though Conjecture 9 is more general than Conjecture 10, one can transform it so as to make it possibly more amenable to calculation, although so far it has proved unassailable.

The properties of a matrix of being proper, or invertible are unaffected if the matrix is multiplied by an invertible matrix or if the diagonal sum with a unit matrix is formed. Using these facts, we can linearize any matrix by enlarging it. (This is sometimes called Higman's trick.) Thus if $c + ab$ occurs in a matrix, in the last row and column say, we can form the diagonal sum with 1 and then carry out the elementary transformations indicated (only the bottom right-hand corner of the matrix is shown):

$$\begin{pmatrix} c+ab & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} c+ab & a \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} c & a \\ -b & 1 \end{pmatrix}.$$

In this way each product can be factorized, and, for example, any matrix over $K_k^* k\langle X\rangle$ can be transformed to the form $A + \Sigma B_i x_i$, where A, B_i are matrices over K. Now we have the following reduction.

THEOREM 9.1. *Let K be a field which is a k-algebra, where k is algebraically closed. Then Conjecture 9 is true if and only if for each K and k, every matrix of the form $I - Ax$ (where $A \in K_n$) over $K_k^* k[x]$, which is improper, is invertible.*

Clearly the condition is necessary; when it holds, any matrix over $K_k^* k[x]$ is invertible or proper, since we can linearize it. Thus we can use induction on the number of indeterminates $x_1, \cdots, x_d$. Write $R = K_k^* k\langle x_1, \cdots, x_d\rangle$, $R' = K_k^* k\langle x_2, \cdots, x_d\rangle$ and let A be a non-invertible matrix over R. We may take A to be of the form $A = A_0 + \Sigma A_i x_i$. If the matrix $B = A_0 + \Sigma_2^d A_i x_i$ (obtained by putting $x_1 = 0$) is non-invertible, it is proper by the induction hypothesis, and hence A is proper. Otherwise we can form $B^{-1}A = I + Cx_1$, where C is a matrix over R'. Now $I + Cx_1$ is invertible over R if and only if Cx_1 is nilpotent (as we see by embedding R in its power series completion). It follows that $I + Cx_1$ is invertible over R if and only if it is invertible over $L_k^* k[x_1]$, where L is the universal field of fractions of R'. But A was non-invertible over R, hence the same holds for $B^{-1}A$, and so $I + Cx_1$ is non-invertible over $L_k^* k[x_1]$, and it is therefore proper (by the hypothesis, because k is clearly *algebraically* closed). It follows that A is proper over R, as claimed.

An affirmative solution of Conjecture 9 would also entail the Freiheitssatz (Conjecture 1, Section 4): let $f \in k\langle x_1, \cdots, x_d \rangle$ involve x_1. Denote by K the universal field of fractions of $k\langle x_2, \cdots, x_d \rangle$. Clearly k is algebraically closed in K (as we see by tensoring with an algebraic closure of k), and f is a non-unit in $K_k^* k[x_1]$, for the only units are products of units in K and $k[x_1]$ (Cohn [60]) and so cannot involve x_1. By Conjecture 9, there is a homomorphism of $K^* k[x_1]$ into a field mapping f to 0. Hence there is a homomorphism of $k\langle x_1, \cdots, x_d \rangle$ which maps f to 0 but is injective on $k\langle x_2, \cdots, x_d \rangle$.

iii) The field spectrum $X(R)$ introduced here is best suited for studying integral domains; for many other quite ordinary rings it may be empty (for example, any matrix ring). To overcome this defect one may consider a larger spectrum $Y = \varinjlim Y_n$, where Y_n is the set (suitably topologized) of all epimorphisms from R to $n \times n$ matrix rings over fields. Again one constructs a sheaf of local rings on Y (cf. Cohn [72[b]]). This construction may be of interest even for n tegral domains such as $k\langle X \rangle$. Thus it now becomes possible to prove that over a free algebra, every improper matrix (relative to Y) is invertible; this shows in particular that every equation over a free algebra has a matrix solution (Cohn [[a]]). An even larger spectrum is the epi-spec described by Bergman [[a]]; it consists of all epimorphisms from R to epimorphism-final rings.

10. Algebraically closed skew fields

A complete discussion of polynomial rings includes not only the function field $k(x_1, \cdots, x_d)$ but also its algebraic closure. Now as is well known, every commutative field k has an algebraic closure $\bar{k}$, characterized up to isomorphism by the properties:

(i) $\bar{k}$ is algebraic over k,

(ii) every equation over $\bar{k}$ has a solution in $\bar{k}$.

In discussing free algebras we therefore turn to the question whether a given skew field has an algebraic closure. In general it will not be possible to satisfy (i); we therefore abandon it and concentrate on (ii). Even here, as we have seen, it is rather difficult to tell whether a given equation has a solution in a suitable extension field, so we modify this condition somewhat. We shall assume (as before) that all our fields contain a given commutative field k as subfield of the centre. A set of equations or inequations ($f \neq 0$) is said to be *consistent* if it can be satisfied in some field extension.

DEFINITION. A field K which is a k-algebra is said to be *existentially closed* over k, if every finite consistent system of equations and inequations in $x_1, \cdots, x_d$ has a solution in K.

Here we can omit the inequations, for if our system is $f_1 = \cdots = f_r = 0$, $g_1 \neq 0, \cdots, g_s \neq 0$, we can introduce new variables $y_1, \cdots, y_s$ and replace the inequations by $g_1 y_1 = \cdots = g_s y_s = 1$.

In a commutative algebraically closed field any consistent set of rational expressions has a solution. This is still true in general, but not quite so obvious (Cohn [75^a]).

THEOREM 10.1. *Let K be an existentially closed field over k, and $A_1, \cdots, A_r$ any set of matrices in $x_1, \cdots, x_d$ which all become singular for certain values of the* x's *in some extension of K. Then there are values of $x_1, \cdots, x_d$ in K for which the matrices become singular.*

It follows from general principles that every field can be embedded in an existentially closed field (essentially because fields possess the amalgamation property, established in Cohn [71^a], cf. Jónsson [56, 60]). However, there will in general be no least existentially closed extension containing a given field, and it is not usually possible to obtain an existential closure which is unique up to isomorphism. In order to achieve uniqueness, A. Robinson [71] has introduced as stronger form of closure, or rather closedness, giving rise to *generic fields. This as* a class of fields with the properties:

(a) every field is embeddable in a generic field,

(b) a subfield of a generic field G is again generic if and only if it is an elementary subfield of G.

A mapping is called *elementary* if it preserves all first order sentences, and a subfield is elementary if the inclusion is an elementary mapping.

Using forcing, Robinson [70, 71] proves that the conditions (a)–(b) determine the class of generic fields uniquely. (Robinson uses absolute fields, that is, he does not single out a subfield of the centre, but it is clear that his results hold in the more general situation.) It has been observed that the generic fields constitute precisely the class of elementary subfields of the homogeneous universal fields constructed by the methods of Jónsson [56, 60]. By contrast the class of existentially closed fields is not closed under elementary equivalence (Wheeler [72]).

The relation between generic and other types of algebraic closure is investigated by Wheeler [72], who remarks that the weaker notion of existential closure is probably sufficient for most uses in algebra. As an illustration, he proves a form of the Nullstellensatz. To state his result he needs to introduce a particular notion of radical of an ideal $\mathfrak{a}$ in a ring R, essentially the intersection of the kernels of all homomorphisms of $R/\mathfrak{a}$ into skew fields. An alternative formulation, using the matrix ideals of Cohn [71[b]], Chapt. 7 is also mentioned by Wheeler; here the radical $\sqrt{\mathscr{A}}$ of a matrix ideal $\mathscr{A}$ is the precise analogue of the commutative notion. Without going into the definitions (Cohn [71[b]], Chapt. 7), we may state the result as follows (Cohn [75[a]]).

THEOREM 10.2. *Let K be an existentially closed field over k, B a matrix and $\mathscr{A}$ a matrix ideal over $K_k^* k\langle x_1, \cdots, x_n\rangle$. Then $B \in \sqrt{\mathscr{A}}$ if and only if B becomes singular at all points over any extension L of K at which the matrices of $\mathscr{A}$ all become singular. If $\mathscr{A}$ is finitely generated, it is enough to consider points over K itself, but this is not true in general.*

The last assertion of Theorem 10.2 is perhaps the most interesting; it depends on the fact that every countable existentially closed field has outer automorphisms (essentially because it has $2^{\aleph_0}$ different automorphisms, cf. Wheeler [72]). Let σ be an outer automorphism and consider the equations $ax = xa^\sigma$ $(a \in K)$, $xy = 1$. They have a solution in a suitable extension of K (for example, in $K(x; \sigma)$), but not in K itself.

Theorem 10.2 means that the "finitely generated" maximal primes of F are associated with the points of K^d, whereas the other maximal primes are associated with points in L^d, for different extensions L of K. In the commutative case such primes do not arise, by the Hilbert basis theorem; so to obtain a parallel theory in the general case, finite generation has to be explicitly postulated.

From another point of view we may say that whereas in the commutative theory every finitely generated field is finitely presented, this no longer holds in general. All this suggests that, in studying the field spectrum of the free algebra, one should confine attention to the finitely generated primes. Perhaps this will lead to finiteness conditions in the field spectrum; such conditions would certainly be needed for a usable dimension theory. However, as Bergman remarks, the notion of a finitely generated prime is somewhat restrictive; as a possible replacement he suggests considering the minimal primes associated with finitely generated matrix ideals. It can be shown that every prime is a directed union of primes of this form, so that there are indeed "enough" such primes to approximate all primes.

In conclusion we list a number of conjectures and problems on free algebras which did not fit into the text.

PROBLEM 11 *Two elements a, b of an integral domain R are said to be* similar *if* $R/aR \cong R/bR$. *Is every similarity class of elements in a free algebra finite (modulo associates)?*

PROBLEM 12 (Bergman). *In a free algebra, is every element similar to a square itself a square?*

PROBLEM 13 *In a free algebra, is the quotient by the ideal generated by a similarity class of atoms (that is, unfactorable elements) necessarily an integral domain?*

PROBLEM 14 *Is the eigenring of a finitely generated non-zero right ideal in a free algebra necessarily commutative?* (For principal right ideals this is the case; cf. [71^b] Prop. 4.6.4, Cor.).

PROBLEM 15 (Lewin [69]). *Is every finitely generated subalgebra of a free algebra finitely presented?*

PROBLEM 16 *The natural mapping* $R \to R_\Sigma$ *is always an epimorphism of rings. Under what conditions on* Σ *is it flat?*

PROBLEM 17 (Bergman). *Does every inversive localization of a prime ring with a polynomial identity satisfy the same identity?* (With *prime* omitted, this is false, as the example of triangular matrix rings shows.)

PROBLEM 18 (Bergman). *For what groups G (or more generally, monoids) is* $k[G]$ *a fir, semifir, 2-fir or (semi)hereditary? Does the answer depend on k?*

PROBLEM 19 (Bergman). *Does every closed subalgebra of* $k\langle\langle X\rangle\rangle$ *meet* $k\langle X\rangle$ *in a free algebra?*

PROBLEM 20 (Bergman). *If* $a \in k\langle X\rangle$ *is a square in* $k\langle\langle X\rangle\rangle$, *is it associated to a square in* $k\langle X\rangle$?

PROBLEM 21 (Bergman). *Given $a \in k\langle X\rangle$, denote by C, C' its centralizers in $k\langle X\rangle$, $k\langle\langle X\rangle\rangle$ respectively. Is C' the closure of C in $k\langle\langle X\rangle\rangle$?*

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